Reprinted from JOURNAL OF APPLIED PHYSICS, Vol. 40, No. 10, 4009–4017, September 1969

Hertzian Fracture Experiments on Abraded Glass Surfaces as Definitive Evidence for an Energy Balance Explanation of Auerbach's Law

F. B. LANGITAN AND B. R. LAWN
School of Physics, University of New South Wales, Kensington, New South Wales, Australia
(Received 6 December 1968)

The question of the physical interpretation of Auerbach's law, that the critical load for production of a Hertzian cone fracture is proportional to the radius of the indenting sphere, has aroused interest in recent years (i) because of its implications concerning the validity of certain brittle fracture criteria and (ii) because of its potential use as a means for measuring fracture surface energies. Two distinct schools of thought, one based on a flaw statistical model and the other on an energy balance concept, have emerged as a result of various attempts to account for this law. This paper describes a theoretical and experimental study aimed at testing the validity of each of these two approaches. Whereas quantitative Hertzian fracture tests have hitherto been extensively made only on surfaces of carefully handled specimens, such as commercial plate glass in the as-received state, in the present case they are made on plate glass surfaces treated by an abrasion process. With as-received surfaces the obscure nature of the distribution of surface flaws, from which the cone cracks initiate, largely precludes a conclusive comparison between the two theoretical approaches, while for abraded surfaces the experimentally justifiable assumption that the flaw distribution is "uniform" leads to simple but widely conflicting predictions from the two approaches, thus providing the basis for a definitive experiment. According to a flaw statistical argument, in conjunction with an empirical critical stress criterion for fracture, Auerbach's law is predicted to break down for tests on uniformly damaged surfaces, with the critical load for cone fracture becoming a sensitive function of the depth of the damage layer. On the other hand, a stepwise application of Griffith's energy balance criterion for fracture to the growth of a cone crack through the inhomogeneous Hertzian stress field predicts strict adherence to Auerbach's law within limits of indenter size, with the Auerbach constant of proportionality in this law being independent of flaw statistics. Static and impact tests, made with steel balls ranging from 0.08 to 1.9 cm in radius on plate glass surfaces abraded with slurries of Nos. 240, 320, 400, and 600 SiC abrasive powder, confirm the essential constancy of Auerbach's law, thereby providing strong evidence for the energy balance explanation. Moreover, it is found that the abrasion treatment leads to a drastic reduction in the scatter in results; as a result variations of less than 10% in the fracture surface energy, which is proportional to the Auerbach constant, should be detectable in experiments performed on a given material under different test conditions.

INTRODUCTION

Hertzian fractures are produced in flat-surfaced brittle solids critically loaded, either statically or dynamically, with a hard sphere. In glasses[1] the cracks have very nearly the shape of a truncated cone, their circular trace on the surface of the indented solid lying in close proximity to the circle of contact between sphere and specimen, while in single crystals[2] crystallographic cleavage somewhat modifies the crack pattern. In 1891 Auerbach[3] demonstrated that the stress level in an indented specimen, just prior to the initiation of a cone fracture, increases with decreasing radius of the spherical indenter. This size effect is an apparent violation of the so-called *critical stress criterion for failure*, which predicts that a specimen will "fail" when a particular stress component reaches a critical value; for brittle fracture the relevant component of stress is the maximum tensile stress in the specimen. This *empirical* failure criterion has had such widespread success in the theoretical treatment of deformation processes in solids (including failure by plastic flow, etc.) that the contrary behavior shown by Hertzian fractures has come to render the Auerbach result an outstanding problem in fracture theory.

Over the years since Auerbach's discovery two schools of thought have evolved from various attempts to explain the Hertzian fracture problem. These are (i) that cone fractures are initiated from incipient flaws present on the specimen surface, and that the critical stress required to make a cone crack propagate spontaneously from a particularly severe

[1] F. C. Frank and B. R. Lawn, Proc. Roy. Soc. (London) **A299**, 291 (1967).
[2] B. R. Lawn, J. Appl. Phys. **39**, 4828 (1968).
[3] F. Auerbach, Ann. Phys. Chem. **43**, 61 (1891).

flaw is governed by the distribution of size and location of such flaws on the surface and (ii) that the formation of a cone fracture is governed by a balance between the strain potential energy released and the surface energy of the crack faces gained as the fracture process proceeds. Both approaches, the flaw statistical argument above and the energy balance condition in its original form as proposed by Roesler,[4] are subject to objections that cannot be satisfactorily discounted. More recently, a theory based on the energy balance concept, but taking into account the details of crack growth through the highly inhomogeneous Hertzian stress field, has been proposed.[1,2] This theory leads to a formal derivation of Auerbach's law from first principles. The details of the two theoretical approaches will be developed in the next section. It is pointed out here that the energy balance concept does *not* reject the proposal that cone cracks develop from surface flaws; it does, however, reject the claim that flaw statistics are the underlying cause of the Auerbach behavior.

Following discussions with other workers in the field[5] the authors have become increasingly aware that the theoretical confirmation of Hertzian fracture data taken from carefully handled, as-received plate glass specimens cannot alone be regarded as constituting absolute proof of the validity of an energy balance mechanism. For all theories require *some* assumption concerning the occurrence of prepresent flaws on carefully handled specimen surfaces, and the very nature of these flaws is barely less obscure now than when they were first hypothesised by Griffith in 1920.[6] In an ideal situation one would like to be able to perform experiments on specimen surfaces whose flaw distribution can be controlled and accurately determined beforehand. One could then aim to design definitive experiments permitting the essential features peculiar to the various Hertzian fracture theories to be subjected to a conclusive test. Experiments on specimens whose surface damage characteristics are predetermined have not hitherto been attempted in the Hertzian fracture test, although more conventional fracture tests have been carried out on brittle specimens mechanically damaged in a controlled manner.

This paper describes one such attempt at a definitive experiment. Controlled damage is introduced into glass surfaces by means of an abrasion treatment; this treatment produces an effectively uniform layer of surface microcracks, whose depth is of the order of the size of abrading particles. The predictions of the two theoretical approaches are then examined in the light of static and impact tests with steel spheres on the glass specimen surfaces. These tests serve to substantiate an energy balance concept of Hertzian fracture.

[4] F. C. Roesler, Proc. Phys. Soc. (London) **B69**, 55 (1956).

[5] One of us (B.R.L.) acknowledges informative discussions with Professor I. Finnie and Dr. H. L. Oh at the University of California, Berkeley and Professor H. Kolsky and Dr. Y. M. Tsai at Brown University on this matter.

[6] A. A. Griffith, Phil. Trans. **A221**, 163 (1920).

THE TWO THEORETICAL APPROACHES TO THE HERTZIAN FRACTURE PROBLEM

Hertzian Contact Theory and Auerbach's Law

Both theoretical approaches mentioned earlier make use of the Hertzian contact equations[7] for a normally loaded spherical indenter on an elastic half-space. With E the Young's modulus and ν the Poisson's ratio of the specimen, and E', ν' the corresponding elastic constants for the indenter material, we may write the radius a of the contact circle as

$$a^3=\tfrac{4}{3}(k/E)Pr, \tag{1}$$

and the distance of mutual approach Z of sphere and half-space from contact as

$$Z=(\tfrac{4}{3}k/E)^{2/3}P^{2/3}r^{-1/3}, \tag{2}$$

where P is the normal load, r is the radius of the sphere, and k is defined by

$$k=\tfrac{9}{16}[(1-\nu^2)+(1-\nu'^2)E/E']. \tag{3}$$

In the Hertzian stress field, which is geometrically similar for all indenter sizes, a is a convenient unit of length, and the mean pressure p_0 between ball and specimen

$$p_0=P/\pi a^2, \tag{4}$$

is a convenient unit of stress.

The Hertzian stress field itself is described in detail in earlier papers.[1,2] We simply point out here some relevent features concerning the distribution of the greatest principal stress (positive values denoting tension) in the loaded half-space. This component of stress attains its maximum at the circle of contact, where it has the value

$$\sigma_m=\tfrac{1}{2}(1-2\nu)p_0. \tag{5}$$

Within the circle of contact it becomes largely compressive, while outside the circle it drops off slowly from its maximum value according to the relation

$$\sigma_R=\sigma_m(a/R)^2, \tag{6}$$

R being the radial distance in the specimen surface from the center of contact. Below the surface outside the area of contact the value of this greatest tensile stress decreases extremely rapidly with depth. One may best picture the indenting process to give rise to a highly tensile "skin" layer in the specimen surface outside the circle of contact.

It is the general aim of a theory of Hertzian fracture to begin from the Hertzian equations (1)–(6), which are valid just prior to cone fracture occurring, and to develop a relation between the critical load P_c and the ball radius r. In particular, one seeks to

[7] H. Hertz, J. Reine Angew. Math. **92**, 156 (1881). Reprinted, in English, in *Hertz's Miscellaneous Papers* (Macmillan and Co., Ltd, London, 1896), Chap. 5.

obtain a theoretical verification of the empirical relation discovered by Auerbach[3],

$$P_c = Ar, \tag{7a}$$

the so-called *Auerbach's law*, A being Auerbach's constant. In an impact test, in which a sphere is released from rest from a critical height h_c on to a horizontal specimen surface, the equivalent aim is to obtain a relation between h_c and r. The equivalent of Auerbach's law (7a) for the impact case can be derived as follows. We suppose that the initial gravitational potential energy of the stationary sphere

$$U_g = \tfrac{4}{3}\pi r^3 \rho g h_c, \tag{8}$$

ρ being the density of the indenter material and g the gravitational acceleration, can, to a first approximation,[8] be equated to the strain potential energy U_s at maximum impression. U_s is expressable as

$$U_s = \int_0^Z P(Z)dZ,$$

so that, from (2), we get

$$U_s = \tfrac{2}{5}(4k/3E)^{2/3} P_c^{5/3} r^{-1/3}. \tag{9}$$

Putting $U_g = U_s$, we obtain

$$P_c = \{\tfrac{1}{3}(10\pi\rho g)^3 (E/4k)^2\}^{1/5} r^2 h_c^{3/5}, \tag{10}$$

with P_c the normal critical force exerted by the decelerating sphere on the specimen. Substituting (10) into (7a) gives

$$r h_c^{3/5} = A\{[3/(10\pi\rho g)^3](4k/E)^2\}^{1/5} = A' \tag{7b}$$

as a statement of Auerbach's law for the impact case.

Flaw Statistical Approach

A common starting point for all flaw statistical theories[9] of fracture is a modified form of the critical stress criterion. The modification takes into account the fact that the stress required to initiate a fracture from a particular flaw depends on the size, location, and orientation of that flaw. For instance, for the special case of a brittle specimen containing a plane, surface flaw of characteristic length c_f oriented perpendicularly to a uniform, uniaxial stress field, Griffith[6] showed that the critical stress required to spontaneously initiate a fracture from the flaw is

$$\sigma_T = [2E\gamma/\pi(1-\nu^2)c_f]^{1/2}, \tag{11}$$

where E and ν are as previously defined and γ is the reversible specific surface energy. Equation (11) is derived from *Griffith's fundamental energy balance equation*

$$\mathcal{G} \geq 2\gamma, \tag{12}$$

which postulates that a crack of unit width will extend when the rate of release of strain potential energy in the system, $\mathcal{G}$, just balances the work per unit area required to create the two new crack surfaces, 2γ. Since Eq. (12) is an expression of the principle of conservation of energy, (11) has a firm physical basis, but it is to be emphasized that the latter equation is derived for the particular case of uniaxial tension: for the more *general* stress state the critical stress criterion does not necessarily follow as a natural consequence from the original Griffith postulate, as will become evident in the next section.

In terms of the initiation of a cone crack in the Hertzian stress field it is accepted that fracture begins from a particularly favorable flaw located close to the circle of contact where the tensile stress is a maximum. By reference to (11), the favorable flaw being hypothetically subjected to a stress state closely approximating uniform uniaxial tension, it may then be argued that the cone crack will spontaneously initiate when σ_m exceeds the critical value σ_T,[10] thereafter to propagate until it finally becomes arrested in the weakening stress field remote from the contact circle. Thus, equating (11) and (5), we have

$$\tfrac{1}{2}(1-2\nu)p_0 = [2E\gamma/\pi(1-\nu^2)c_f]^{1/2}.$$

We may express p_0 in terms of the more readily measured parameters P and r by eliminating a from (1) and (4); substituting the resulting expression into the equation above and rearranging, the critical condition for fracture becomes

$$\frac{P_c}{r^2} = \left\{\frac{256\sqrt{2}\pi^{3/2}k^2\gamma^{3/2}}{9(1-\nu^2)^{3/2}(1-2\nu)^3 E^{1/2}}\right\} c_f^{-3/2}, \tag{13a}$$

with P_c the critical load.[11]

Thus if all cone cracks were to originate from flaws of the same size we would expect to find a relation of the kind $P_c \propto r^2$. The essence of the flaw statistical argument is that indenters of larger radius r sample a larger area of specimen surface, so that for a typical flaw distribution on, say, an as-received plate glass

---

[8] We neglect energy losses owing to air resistance and to dissipative causes during the impulse. An obvious justification of this approximation is the observation that the sphere rebounds to very nearly its original height. Hertz, in his original paper, also argued that the duration of impact is sufficiently great that the quasistatically determined stress field is a good approximation in the impact case.

[9] See, for instance, A. S. Argon, Proc. Roy. Soc. (London) **A250,** 482 (1959); E. W. Sucov, J. Amer. Ceram. Soc. **45,** 214 (1962); Y. M. Tsai and H. Kolsky, J. Mech. Phys. Solids **15,** 29 (1967); H. L. Oh and I. Finnie, J. Mech. Phys. Solids **15,** 401 (1967).

[10] Some statistical theories take into account the experimental observation that the cone cracks often form outside the circle of contact, presumably from a particularly severe flaw. In this instance it would be more appropriate to expect initiation of fracture when σ_R [Eq. (6)] at the location where the severe flaw exceeds σ_T. With our experiments, as will be pointed out later, the cone cracks initiate close to the circle of contact, so that this extra complication can be ignored here.

[11] It is true that not all Hertzian fracture theories based on flaw statistics necessarily assume that Eq. (11) in its specific form is applicable to the Hertzian stress field. However, all such theories inherently require that fracture will spontaneously initiate when the tensile stress acting on a severe flaw reaches *some* critical level, and that this stress level will depend on the size of the flaw, so that a relation of the form $P_c/r^2 = f(c_f)$ will still arise, and the basis of the argument presented here will not be altered in any way.

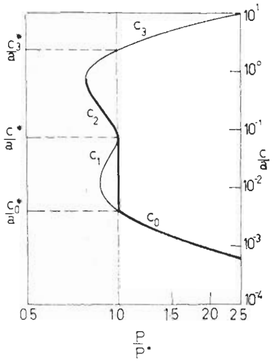

FIG. 1. Plot of c/a as function of P/P^* for downward cone crack extension within Hertzian stress field (full, light curve). Heavy curve indicates indenter load necessary to initiate full cone crack from flaw (see text). Asterisk denotes value of a variable when $P=P^*$. Curve calculated for $\nu=\frac{1}{3}$.

specimen, the chance that a particularly dangerous flaw be encompassed within a region of high tension increases with indenter size. Thus, on the average, one might expect cone cracks to initiate from more severe flaws as the indenter size is increased, so that the magnitude of c_f effectively increases with r, in which case P_c in (13a) would increase less rapidly than with r^2, in qualitative accordance with Auerbach's observation.

There are two objections[4] to the flaw statistical concept. First, it is difficult to conceive of a distribution of flaws always having just the right statistics to account for such a simple power law as (7a), especially as this law holds for a wide variety of glasses[12] and single crystals.[2] Secondly, since smaller indenters sample smaller areas of specimen surface, one would also expect the scatter in results to increase with decreasing r. This is not observed.

The same arguments can be applied to the impact test. Making use of Eq. (10) the critical condition (13a) for fracture becomes

$$h_c=\left\{\left(\frac{256\sqrt{2}}{9}\right)^{5/3}\left(\frac{48^{1/3}k^4}{10\pi\rho g E^{3/2}}\right)\left[\frac{\pi\gamma}{(1-\nu^2)(1-2\nu)^2}\right]^{5/2}\right\}c_f^{-5/2}. \tag{13b}$$

Energy Balance Approach

Roesler[4] pointed out that Auerbach's law follows naturally from the Hertz equations if one makes the assumption that the small fraction of prior elastically stored energy released by the growth of a cone crack be size independent, the *total* loss in strain energy just balancing the *total* gain in surface energy of the new crack surfaces. The major objection raised to Roesler's assumption is that, unlike Griffith's energy balance postulate [Eq. (12)], it has no sound physical basis. The assumption implies reversibility in the entire crack growth, which is contrary to the observation that cone cracks propagate with nonzero velocity.

Such difficulties are overcome by applying Griffith's equation (12) in a stepwise manner to the growth of the crack from a flaw through to its ultimate completion. The $\mathcal{G}$ term can be computed, in an approximate manner, using the techniques of Irwin's *fracture mechanics*[13] as a function of the reduced crack length c/a. For the downward-extending cone crack in an isotropic solid Griffith's equation in equilibrium form becomes[1,2]

$$P=\frac{2\pi^3 kr\gamma}{3(1-\nu^2)}\left\{\left(\frac{c}{a}\right)^{1/2}\int_0^{c/a}\frac{[\sigma(b)/p_0]d(b/a)}{(c^2/a^2-b^2/a^2)^{1/2}}\right\}^{-2}, \tag{14}$$

b being the distance measured along the downward-extending crack path and $\sigma(b)$ the *prior* normal stress along this path. The dimensionless quantity within the curly bracket may be numerically integrated on a computer, thus permitting P to be computed as a function of c/a. The inhomogeneous nature of the stress field has a profound effect on the function $P(c/a)$, as is seen in Fig. 1. The "hump" in the curve is of special significance (thus the asterisk notation, signifying values of any variable corresponding to $P=P^*$ at the hump), and is directly attributable to the presence of the tensile "skin" mentioned earlier. A full interpretation of the curve in Fig. 1 has been given elsewhere[1,2] and only a brief indication of how one may predict the growth of the cone crack from an incipient flaw will be presented here.

The flaw will first tend to form a surface ring around the indenter,[1,2] and this ring will subsequently advance downward into the specimen. In order to satisfy Eq. (14) for downward extension of a surface ring crack of depth c_f the load on the indenter must be made to exceed the value corresponding to the point $(P/P^*, c_f/a)$ on the equilibrium curve. The crack will then extend in a stable or unstable manner according to whether the slope of the curve is positive or negative at that point. Auerbach's law follows if c_f lies within the range $c_0^*\leq c_f\leq c^*$ when the Griffith equilibrium condition (14) first becomes satisfied. If the first equilibrium occurs on the c_0 branch a small increase in load will cause unstable extension to the c_1 branch at constant load. If the first equilibrium occurs on the c_1 branch, however, an incremental increase in load will cause an incremental extension of the ring crack along the equilibrium curve. Ultimately, therefore, regardless of the initial size of the incipient flaw (so long as it falls within the special size range $c_0^*\leq c_f\leq c^*$), the cone crack will grow along the c_1, branch, i.e., it will grow downward from the specimen surface in a stable manner as the indenter load is gradually increased. Events up to this point will generally pass unnoticed since $c^*\simeq 0.075a$ represents the maximum attainable depth of the surface ring before it becomes unstable again and proceeds to c_3. It is this final phase of development to c_3 which is observed in an experiment, and the condition for it to happen is that

$$P_c\geq P^*\simeq 4\times 10^5 kr\gamma, \tag{15a}$$

the right hand side of (14) being evaluated for $c^*/a=$

[12] J. P. A. Tillet, Proc. Phys. Soc. (London) **B69**, 47 (1956).

[13] G. R. Irwin, *Handbuch der Physik* (Springer-Verlag, Berlin, 1958), Vol. 6, p. 551.

0.075 (Fig. 1) and $\nu=\frac{1}{3}$. Thus, the equilibrium form of this equation may be written

$$P_c/r \simeq 4\times10^5 k\gamma = A, \qquad (c_0 \leq c_f \leq c^*) \qquad (15b)$$

which is Auerbach's law. We note that the approach to the Auerbach condition is one of *reversible* growth along the c_1 branch. Also, Auerbach's constant A is *completely independent of the original flaw size* c_f, all flaws hypothetically growing into a surface ring of depth c^* before the critical load P^* is exceeded. This explains the observed reduction in scatter in results in the region where Auerbach's law becomes obeyed. For flaws outside the range $c_0^* \leq c_f \leq c^*$, however, Auerbach's law will break down,[1] the cone crack propagating unstably to its final length c_3 from the c_0 or c_2 branches without any intermediate intersections with the c_1 branch. The critical load required to produce the cone crack is clearly dependent on the value of c_f in this case, and is therefore subject to any statistical variation in the distribution of flaws.

From Fig. 1 we can deduce P_c as a function of ball radius r over a wide range of r. This is feasible since a increases with r, so that the reduced flaw length c_f/a should correspond to a lower ordinate value in Fig. 1 as the indenter size increases; on this basis we might expect a transition from the Auerbach behavior outside a certain range of indenter size. To show this more formally we write

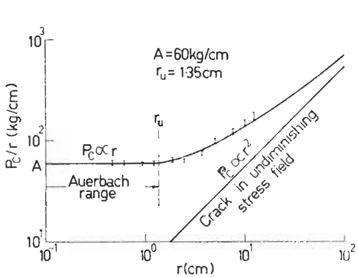

FIG. 2. Plot of P_c/r as function of r. Data points taken from Fig. 3 of Tillet's paper.[12] Upper curve is generated from heavy curve in Fig. 1 and is fitted to the data by adjusting parameters A and r_u (see text). Lower curve calculated for a hypothetical crack in a uniform tensile stress field (σ_m).

$$P_c/r = AP_c/P^*, \qquad (16)$$

noting that $P^*=Ar$ from (15). The critical load P_c necessary to make a surface ring of depth c_f extend to c_3 is indicated by the heavy line superimposed on to the curve in Fig. 1. We also rewrite (1), at critical loading, in the form

$$c_f^3(a/c_f)^3 = (\tfrac{4}{3}k/E)(P^c/P^*)Ar^2. \qquad (17)$$

Associating the special value $r=r_u$ with the point $(1, c_0^*/a)$ in Fig. 1, (17) becomes, with $c_f/a=c_0^*/a\simeq 4.0\times10^{-3}$,

$$c_f^3(1/4.0\times10^{-3})^3 = (\tfrac{4}{3}k/E)Ar_u^2. \qquad (18)$$

Dividing (18) into (17) yields

$$r = r_u\{[4.0\times10^{-3}/(c_f/a)]^3[1/(P_c/P^*)]\}^{1/2}. \qquad (19)$$

From points $(P_c/P^*, c_f/a)$ along the heavy line in Fig. 1 we can generate an equivalent plot of P_c/r against r using (16) and (19). Such a plot is shown in Fig. 2. The two parameters A and r_u are here adjusted to give a best fit of the theoretical curve with data taken from fig. 3 of Tillet's[12] paper. Thus the present theory appears to be capable of explaining not only Auerbach's law, but also the behavior outside the range of validity of Auerbach's law. Also included in Fig. 2 is the computed curve corresponding to the behavior that would be expected if the cone crack were to propagate through a tensile stress field undiminishing from the value σ_m at the circle of contact; this curve corresponds to the flaw statistical equation (13a). It is readily appreciated from a comparison between the two theoretical curves that the higher stress gradients which exist when smaller indenters are used have a significant effect in increasing the critical load necessary to propagate a cone crack from a surface flaw.

TABLE I. Predicted Hertzian fracture relations for specimens with a uniform distribution of surface flaws.

	Static	Impact
Flaw statistical	$P_c/r^2 \propto c_f^{-3/2}$	$h_c \propto c_f^{-5/2}$
Energy balance	$P_c/r=$const	$rh_c^{3/5}=$const

Just as we expect a transition in behavior from $P_c\propto r$ to $P_c\propto r^2$ in the static test we can, by comparing (7a) with (7b), and (13a) with (13b), predict an analogous transition from $h_c\propto r^{-5/3}$ to $h_c=$const in the impact test.

Application to the Special Case of a Specimen With a Uniformly Damaged Surface

If we can mechanically damage brittle specimens in such a way that c_f is effectively constant over the entire test surface we can devise a particularly simple test to distinguish between the predictions of the two theoretical approaches outlined above. From the flaw statistical approach we would then predict, for a given c_f, the fracture relations $P_c/r^2=$const and $h_c=$const [Eqs. (13a) and (13b)]; the constants in these relations are, however, both sensitive functions of c_f, so that c_f may be used as a test variable. From the energy balance approach we would still predict adherence to Auerbach's law, $P_c/r=$const or $rh_c^{3/5}=$const [Eqs. (15), (7a), and (7b)], within a certain

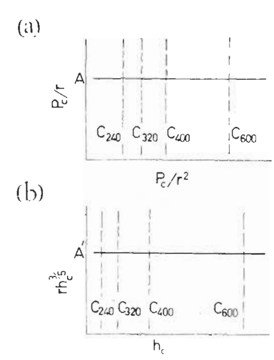

FIG. 3. Theoretically predicted curves for (a) static test, (b) impact test, on uniformly damaged surfaces. The horizontal full line represents the energy balance prediction (Auerbach's law) and is independent of c_f. The vertical broken lines represent the flaw statistical prediction for various c_f (plotted for values of c_f in Fig. 6).

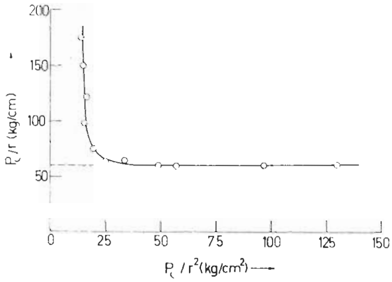

FIG. 4. Plot of Tillet's data[12] for as-received plate glass. Note transition from $P_c \propto r$ to $P_c \propto r^2$ (approximately) corresponding to r exceeding r_u.

range of indenter size; in this case the constants in the two relations are independent of flaw size c_f. We summarize the predicted relations in Table I and plot them schematically in Fig. 3. Thus a plot of P_c/r against P_c/r^2 for a static test, and $rh_c^{3/5}$ against h_c for an impact test, using c_f as an independent variable, should distinguish clearly between the two sets of theoretical predictions. If we venture outside the range of validity of Auerbach's law as specified by the energy balance theory, the horizontal straight lines in Fig. 3 should exhibit a "tail," as demonstrated in Fig. 4. This latter figure, plotted from Tillet's[12] data for as-received glass plate, tails off corresponding to the transition from $P_c \propto r$ to (very nearly) $P_c \propto r^2$; the location of the transition point $(P_c/r_u^2, A)$ will, since r_u increases with $c_f^{3/2}$ [Eq. (18)], move further to the left on this plot as more severely damaged specimen surfaces are tested.

EXPERIMENTS ON ABRADED GLASS SURFACES

Introduction of the Damage

The grinding of brittle solids with abrasive particles has long been known to introduce an intersecting network of microcracks into the treated surfaces.[14] Mould and Southwick,[15] in a series of experiments on the fatigue properties of glass, found abrasion techniques ideally suited to varying the fracture strength of glass microscope slide test specimens; moreover, by virtue of the uniformity of surface damage thus produced, the abraded specimens showed a marked reduction

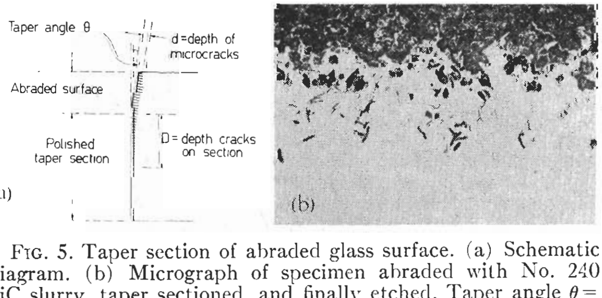

FIG. 5. Taper section of abraded glass surface. (a) Schematic diagram. (b) Micrograph of specimen abraded with No. 240 SiC slurry, taper sectioned, and finally etched. Taper angle $\theta = 1/7$ rad. Width of field 1 mm.

[14] F. W. Preston, Trans. Opt. Soc. **23**, 141 (1922).

[15] R. E. Mould and R. D. Southwick, J. Amer. Ceram. Soc. **42**, 582 (1959).

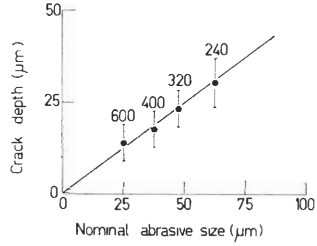

FIG. 6. Depth, c_f, of most severe microcracks on abraded glass surfaces as function of nominal particle size. Error bars indicate maximum variation observed (see text).

in the scatter in fracture strength. Various workers have made numerous attempts, using different metallographic techniques, to measure the depth of abrasion damage in glasses and brittle single crystals. The severity of the damage is found to be dependent on the specific abrasion conditions; generally, however, it appears that the depth of microcracking below an abraded specimen surface is always a little less than the nominal size of the abrading particles.

In our experiments the most satisfactory results were obtained by abrading specimen surfaces with a slurry of loose abrasive powder and water placed on a smooth glass slab. By using either No. 600, 400, 320, or 240 SiC abrasive (nominal particle sizes 25, 37, 47, and 62 μ, respectively) four test surfaces, representing increasing severity of surface damage, could be prepared in this way. The depth and distribution of microcracks in such glass specimens have been examined by the standard metallographic technique of taper sectioning. Figure 5(a) is a schematic representation of the technique: the true depth d of microcracks is measurable as $d \simeq D\theta$, where D is the apparent depth of the cracks on the polished taper section and θ is the taper angle. Figure 5(b) shows a micrograph of a specimen abraded with No. 240 abrasive; the taper angle is $\theta = 1/7$ rad, and the specimen has been etched in a solution of 5% HF for 30 sec to reveal the pattern of microcracks. The surface profile is seen as the boundary between the abraded surface (dark, crazed region at top of micrograph) and the polished section (light background). In estimating the degree of surface roughness and the depth of the cracks extending into the specimen it is to be remembered that distances are exaggerated sevenfold in a vertical direction on the polished section. Figure 5(b) serves to demonstrate the uniformity of surface damage over distances in the specimen surface large compared to the mean separation of microcracks. Thus, in terms of a cone crack experiment, we may regard the damage as uniformly distributed if the circumference of the circular trace of

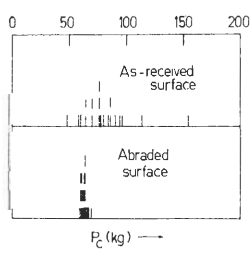

FIG. 7. Scatter in the critical load required to produce cone fractures on an as-received glass surface (top) and on the same surface subsequently abraded with No. 400 SiC slurry (bottom). Twenty tests made on each surface with ball $r = 0.635$ cm.

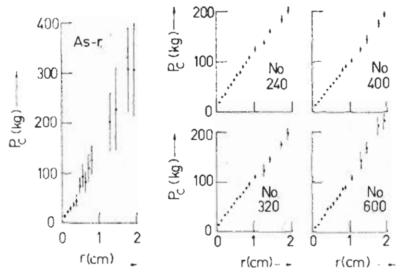

FIG. 8. Plots of P_c against r for as-received (As-r.) and abraded glass surfaces. Error bars denote standard deviations about mean values.

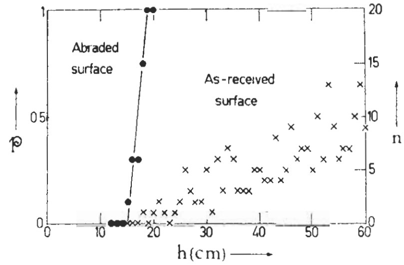

FIG. 10. Probability of producing a cone fracture (number of observed fractures, n, expressed as a fraction of 20 attempts) as a function of height of fall of steel sphere, $r=0.635$ cm, onto 1 in. plate glass. ×: as-received surface, ●: surface subsequently abraded with No. 400 abrasive.

the cone crack on the specimen surface greatly exceeds this mean separation distance. The width of the field in Fig. 5(b) is 1 mm, and the circumference of the smallest cone crack measured in the indentation experiments described in the next section just exceeds this length, so that even for the most severe surface damage the above condition for uniformity is satisfied. Further evidence supporting this claim to uniformity of damage will be presented in the following section.

Despite the overall uniformity of the damaged layers over large distances in abraded surfaces there is some uncertainty associated with the absolute depth c_f of the most severe flaws. This is largely owing to variations in the profiles of the abraded surfaces and in the depths of the deepest cracks, over small distances (<1 mm) across the taper sections. Taking these factors into account the values of c_f have been estimated for each of the four test surfaces. The results are plotted in Fig. 6; it is seen that within the accuracy of the data the microcrack depth increases linearly with nominal abrasive size. The vertical lines in Fig. 3 have, in fact, been plotted on the basis of the "linearity" shown in Fig. 6, i.e., by taking c_f corresponding to the four abrasive sizes to have relative values $c_{600}:c_{400}:c_{320}:c_{240}=25:37:47:62$.

Static and impact ball indentation tests have been made on plate glass[16] specimens abraded as above. Slabs of $\frac{1}{2}$-in. plate, side length 2 in. were used for the static tests, and 1-in. plate, side length 6 in., for the impact tests. Over 100 tests could be made on any one slab.

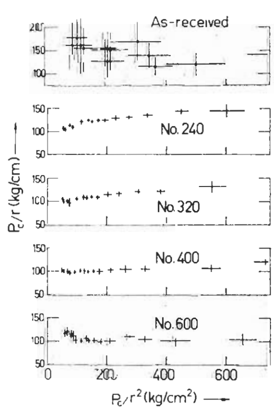

FIG. 9. Plot of P_c/r against P_c/r^2 for static tests on as-received and abraded glass surfaces. Error bars denote standard deviations.

[16] Pilkington plate glass (Pilkington Australia).

Static Tests

The static indentation tests were carried out by mounting steel spheres into the underside of an Instron testing machine crosshead and by then driving the crosshead downward so that the spheres loaded normally onto glass specimens seated on a compression load cell. The sudden initiation of a cone crack could be viewed through the sides of the glass slabs and the critical load at this point duly noted. For the viewing of this sudden crack growth it was found advantageous to operate the crosshead at minimum speed (0.02 cm/min for the particular Instron model available to us) and to carefully illuminate the glass specimen with a microscope lamp. For a given indenter size and a given glass specimen the critical load varied according to the specimen surface preparation. Figure 7 illustrates the spread in P_c for 20 tests made on each of two specimen surfaces with a steel ball of radius $r=0.635$ cm. The upper set of data in Fig. 7 represents a carefully handled, as-received glass surface, and the lower set of data represents the same surface subsequently abraded; it is clear that the treated surface shows reduced scatter, in accordance with the hypothesis that the abrasion process introduces a uniform layer of flaws into the specimen surface.

Similar data have been collected for tests made with a number of steel spheres within the size range $r=0.08-1.9$ cm on both as-received and abraded surfaces. From these data the mean values of the critical loads and their standard deviations have been computed, using an average of 20 tests per ball for the

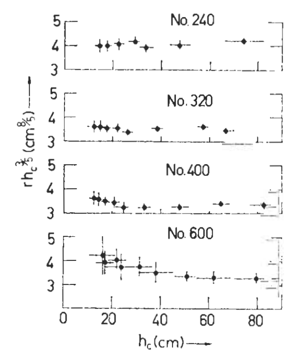

FIG. 11. Plot of $rh_c^{3/5}$ against h_c for impact tests on abraded glass surfaces. Error bars denote standard deviations.

as-received glass surfaces and 10 tests per ball for each of the four abraded surfaces. The results are plotted in Fig. 8. Each plot can be represented closely by a straight line passing through the origin, although this representation is not exact for some of the cases in Fig. 8. More useful information is obtained by replotting the data on a graph of P_c/r against P_c/r^2 (Fig. 9) and by comparing the results with the theoretically predicted curves in Fig. 3(a). From such a comparison two features emerge in support of the energy balance concept of Hertzian fracture. (i) For all surfaces tested the data appear to obey closely Auerbach's law [Eq. 7(a)]. As qualification to this statement it is pointed out that the plot for the as-received and No. 600 surfaces shows evidence for the beginnings of a transition of the type depicted in Fig. 4, while the more severely abraded surfaces (Nos. 240 and 320) show evidence of a slight departure from horizontal straight line behavior of less certain origin (see below). (ii) The mean values of Auerbach's constant A show only slight systematic variation over the range of c_f represented by the four abraded test surfaces. Moreover, any such variation as does exist contrasts with the trend predicted by the flaw statistical relation for static tests given in Table I. The observation that slightly higher values of P_c are required to initiate cone fractures on the more severely abraded surfaces (see Fig. 8) is, in fact, a direct repudiation of the generality of the flaw statistical assumption that fractures are more readily propagated from more severe flaws.

Thus, apart from the small systematic departures mentioned above the results of the static indentation tests are in accord with the predictions of the energy balance theory outlined in a previous section. The reason for the departures from ideal behavior is not known with certainty, but a possible explanation is as follows. It is a well-known fact that mechanical abrasion leaves brittle surfaces in a state of residual compression.[17] This compression will tend to oppose the early stages of crack growth, particularly for small indenters on severely abraded surfaces. In terms of the Frank and Lawn theory we would expect any such effect to become significant when the depth of damage, c_f, becomes comparable with the depth of the stable surface ring crack, c^*, just prior to the propagation of a cone fracture. From Fig. 1, $c^*\simeq 0.075a$, so that, in conjunction with Eq. (1), we calculate c^* to lie within the range 8–60 μ for r within the range 0.08–1.9 cm. Thus for the No. 240 surface, with $c_f\simeq$ 30 μ (Fig. 6), we might anticipate systematic increases in P_c for all but the largest indenters, while for the No. 600 surface, with $c_f\simeq 12$ μ, only the very smallest indenters would be expected to show a similar effect. This is in qualitative agreement with the trends shown by the data in Figs. 8 and 9. It is also likely that the surface roughness caused by the abrasion process modifies locally the Hertzian stress field in the near vicinity of the area of contact. The possible influence of surface roughness on P_c is more difficult to assess, although it is again likely to be most important for severely abraded surfaces {e.g., the maximum variation in surface profile for the No. 240 surface is of the order of 10 μ [Fig. 5(b)], which is comparable with the smallest values of c^*}, and for small indenters.

[17] See, for instance, F. Twyman, *Proceedings of the Optical Convention* (Norgate and Williams, London, 1905), p. 52; A. J. Dalladay, Trans. Opt. Soc. **23**, 170 (1922); W. C. Dash, J. Appl. Phys. **29**, 228 (1958); F. C. Frank, B. R. Lawn, A. R. Lang, and E. M. Wilks, Proc. Roy. Soc. (London) **A301**, 239 (1967).

Impact Tests

Impact tests were carried out by allowing steel spheres within the radius range $r=0.2$–0.8 cm to fall freely under gravity onto the specimen surfaces. Each sphere was released 20 times from an electromagnet fixed at each of a series of preselected heights above the specimen surface, and the probability of fracture $\mathcal{P}$ (i.e., proportion of fractures out of 20 drops) thus recorded as a function of height. Figure 10 shows typical data obtained by dropping a ball of radius $r=0.635$ cm onto an as-received surface and onto the same surface subsequently abraded (cf. Fig. 7). Again it is clear that the abrasion process leads to a marked reduction in scatter. An examination of the relative locations of circle of contact (made visible by lightly smearing grease on to the indenting sphere) and surface trace of the cone cracks revealed a further contrast in behavior between as-received and abraded surfaces; for the as-received surface the cone cracks consistently formed well outside the circle of contact, whereas for the abraded surface the surface ring crack always very nearly coincided with the contact circle. This constitutes further evidence for the state of relative uniformity of damage on the abraded surface.

From distribution curves such as shown in Fig. 10 the critical height h_c is calculated by statistical means as the mean height corresponding to a 50% probability of fracture. The functional relation between the values of h_c thus determined and r for each of the four abraded test surfaces is represented in Fig. 11 by means of a plot of $rh_c^{3/5}$ against h_c. Comparing Fig. 11 with the theoretical curves in Fig. 3(b) we see again that Auerbach's law [Eq. (7b)] is closely obeyed in each case and that Auerbach's constant is barely affected by the severity of the abrasion treatment, with only small deviations from ideal behavior analogous to those observed in the static tests. Thus the impact data are also in accord with the energy balance relations in Table I.

CONCLUSIONS

Compelling evidence for an energy balance explanation of Auerbach's law has been provided by

Hertzian fracture experiments on abraded glass surfaces. The weight of this evidence depends on the assumption that the abraded test surfaces contain a "uniform" distribution of flaws. This assumption receives experimental justification from (i) an examination of taper sections of abraded specimen surfaces, in which the mean spacing between flaws is small compared with the circumference of the contact circle; (ii) the observation that cone cracks formed on abraded surfaces appear to initiate at some location very close to the contact circle; and (iii) the observation that the scatter in results is markedly reduced when as-received glass surfaces are subjected to the abrasion treatment. The basic starting point of the flaw statistical approach, that a cone crack initiates spontaneously from a particularly severe flaw when a critical tensile stress is reached, is incompatible with two features of the experimental data. The first of these features is the close obeyance of Auerbach's law. The central argument in the statistical explanation of Auerbach's law is that larger indenters, which produce larger cone cracks, stand a greater chance of encompassing a severe flaw within the highly stressed region close to the circle of contact. While this argument has plausibility when applied to as-received glass surfaces, on which the surface flaws may be considered to be sparsely distributed, it breaks down when applied to surfaces containing a high density of evenly distributed flaws. The second feature is the lack of influence of flaw size on the critical loading required to initiate a cone fracture. One would expect more severely abraded surfaces to fracture more easily; if anything the reverse trend is observed in our experiments. These two features of the Hertzian fracture data are, on the other hand, quite consistent with an energy balance theory which considers in detail the special nature of the inhomogeneous Hertzian stress field. A major prediction resulting from this theoretical approach, namely, that Auerbach's law be independent of flaw distribution (so long as the indented surface contain a sufficient density of flaws within a certain size range), is thereby confirmed, notwithstanding the small systematic deviations mentioned in the previous section.

Apart from investigating the theoretically predicted dependence of the critical indenter load on ball radius and flaw size there exists, in principle, another, more direct method for distinguishing between the two approaches to the Hertzian fracture problem. This method concerns the manner in which the ultimately completed cone crack grows from the initial surface flaw. The flaw statistical postulate is that the cone crack develops spontaneously from the flaw at critical load, whereas the energy balance theory of Frank and Lawn[1] predicts that, within the limits of validity of Auerbach's law, the flaw first grows into a stably propagating surface ring before critical loading is attained. In our experiments the only means of observing the growth of the cone cracks was by visual inspection through the side walls of the indented glass specimens; no evidence for the existence of a stable surface ring crack was obtained from these observations. However, the maximum possible depth of such a ring crack, c^*, is only of the order of 60 μ for a ball of $r=2$ cm and it is doubtful that cracks of this depth would be easily detected. A sensitive optical arrangement is at present being designed in this laboratory for the express purpose of viewing the initiation stages of cone crack growth in greater detail.

It has been pointed out in previous papers[1,2] that Auerbach's law has potential importance in that Auerbach's constant A (or A') is proportional to the fracture surface energy γ in the energy balance equation (15b). Thus, by recording A as a function of some test variable of interest (temperature, say), one can follow any variations in the γ term. Apart from other attendant advantages unique to the Hertzian fracture test[1,2] the present results indicate that this method for investigating surface energy variations should be capable of unusually high accuracy if the test conditions are properly chosen. For instance, with regard to the present data the combination of smallest scatter in results and least systematic deviation from Auerbach's law is obtained for tests made on specimens abraded with No. 400 SiC slurry. Thus, for 12 balls within the size range $r=0.5$–1.9 cm we calculate a mean value of $A=99.8\pm2.2$ kg/cm from the static test data on the No. 400 specimens, while for nine balls within the size range $r=0.2$–0.8 cm we similarly calculate $A'=3.40\pm0.13$ cm$^{8/5}$ from the impact data. It should therefore be possible to detect variations in γ of only a few percent by this method.

ACKNOWLEDGMENTS

We wish to thank P. Holmes for technical assistance with the indentation equipment and D. Robertson for discussions on the statistical processing of the data. The Schools of Textile Technology and Metallurgy kindly made available to us their Instron machines, on which the static indentation data reported here were obtained. One of us (F.B.L.) acknowledges financial assistance from a Commonwealth Government Grant under the Colombo Plan.